BALANCED OPTIMAL SATURATED MAIN EFFECT PLANS OF THE 2^n FACTORIAL AND THEIR RELATION TO (v,k,λ) CONFIGURATIONS

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by

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Abstract

This paper provides a characterization of balanced saturated main effect plans of the 2^n factorial in terms of D'D rather than X'X, where D is the $(n + 1) \times n$ treatment combination matrix and X is the $(n + 1) \times (n + 1)$ design matrix. This characterization is made possible utilizing a simple matrix transformation of X to a (0,1)-matrix. Besides this result, optimal (in the sense of maximum determinant of X'X) balanced saturated main effect plans of the 2^{4m-1} factorials are discussed utilizing optimality theorems of (v,k,λ)-configurations. Also, some optimality results are given using complementary (v,k,λ)-configurations and the corresponding complementary main effect plans of the 2^{4m-1} factorial.

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1. Summary. In a recent paper Raktoe and Federer [1970a] presented a method of obtaining the information matrix for a saturated main effect plan of the 2^n factorial directly from the treatment combination matrix D. This result allowed a characterization of certain classes of optimal plans in terms of D'D rather than X'X. The process which made this characterization possible depended upon a non-singular matrix G through which the normal equations could be modified as a function of D alone. The present paper first establishes an equivalent method of relating X'X to D'D. We then characterize the class of balanced saturated main effect plans in terms of D'D and finally present some results on optimal saturated main effect plans. In this last case the connection between the number of +1's in these plans and v,k,λ configurations is also brought out.

2. Introduction. Fractional factorial designs present some of the most challenging problems in treatment designs. Even when dealing with the simplest situation, such as main effect plans of the 2^n factorial, one is confronted with problems of a highly complex combinatorial nature. Some of these problems have been pointed out and investigated by Federer, Paik, Raktoe, and Werner [1972], Paik and Federer [1970, 1972], and Raktoe and Federer [1970b, 1971]. Problems for

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other types of plans from the 2^n factorial are currently being studied intensively by Srivastava and Chopra [1971], Srivastava and Anderson [1970], Banerjee [1970], and Srivastava, Raktoe, and Pesotan [1971]. All of these authors have demonstrated the mathematical and statistical richness of factorial experiments.

To make this paper relatively self-contained we introduce the following notations and definitions:

(i) In a 2^n factorial experiment with n factors at two levels each, a treatment combination is an n-tuple $(x_1, x_2, \cdots, x_n)$, with $x_i \in \{0,1\}$.

(ii) A set of $(n + 1)$ treatment combinations arranged in arbitrary order in an $(n + 1) \times n$ matrix D (a row being a treatment combination) with the aim to estimate the vector β consisting of the mean and the main effects is called a saturated main effect plan.

(iii) The $(n + 1) \times (n + 1)$, $(-1,1)$-matrix X corresponding to D and the parameters in (ii) is called the design matrix of D; $X'X$ is called the information matrix of the design D.

(iv) I will be a square identity matrix, J a rectangular matrix consisting of +1's, and $\mathbf{1}$ will be a column vector of +1's; 0 is either a matrix or vector of 0's.

(v) A balanced saturated main effect plan of the 2^n factorial is a design D such that: (a) each element of β (= vector consisting of the mean and the main effects) is estimated with the same variance, (b) the covariance between the estimates of the mean μ and a main effect is a constant, and (c) the covariance between estimates of two main effects is another constant.

(vi) Optimality of a saturated main effect plan may be defined in many ways. We will indicate whether we are using maximum determinant of $X'X$ or minimum trace of $(X'X)^{-1}$ or maximum root of $(X'X)^{-1}$ as a criterion.

3. Relation between X'X and D'D. In the paper by Raktoe and Federer [1970a], it was shown that through the transformation

$$(3.1)\qquad G = \tfrac{1}{2}\left[\begin{array}{c|c} 2 & \mathbf{1}' \\ \hline 0 & I \end{array}\right]$$

one obtains $XG = X^*$ such that:

$$(3.2)\qquad X^* = [\mathbf{1} \mid D]$$

$$(3.3)\qquad \det G = 2^{-n}$$

$$(3.4)\qquad G^{-1} = \left[\begin{array}{c|c} 1 & -\mathbf{1}' \\ \hline 0 & 2I \end{array}\right]$$

$$(3.5)\qquad X = X^*G^{-1}$$

$$(3.6)\qquad \det X^* = 2^{-n} \det X \text{ or } \det X = 2^n \det X^*$$

$$(3.7)\qquad X'X = \left[\begin{array}{c|c} n+1 & z' \\ \hline z & Z \end{array}\right], \text{ where}$$

$$z = -(n+1)\mathbf{1} + 2D'\mathbf{1}$$

$$Z = (n+1)J - 2D'J - 2J'D + 4D'D.$$

We now show that it is not necessary to use G.

Theorem 3.1. *If a J matrix is added to X and this sum is multiplied by* $\tfrac{1}{2}$ *then the resultant matrix* X^{**} *is such that:*

$$(3.8)\qquad \tfrac{1}{2}[X + J] = X^{**} = [\mathbf{1} \mid D]$$

$$(3.9)\qquad X^{**} = XG = X^*$$

$$(3.10)\qquad X'X = \text{as in } (3.7).$$

Proof. Part (3.8) follows directly from the fact that addition of a J-matrix to X changes -1's of X to 0's and the +1's to 2's, so that multiplication by $\frac{1}{2}$ results in a matrix with the first column consisting entirely of +1's (since X has its first column equal to +1's), and the rest of the elements being 0's and +1's. The proof of (3.9) follows immediately from (3.2). Finally, the proof of part (c) is obtained as:

$$
\begin{aligned}
X'X &= [2X^* - J]'[2X^* - J] \\
&= 4X^{*\prime}X^* - 2X^{*\prime}J - 2J'X^* + J'J \\
&= 4[\mathbf{1} \vdots D]'[\mathbf{1} \vdots D] - 2[\mathbf{1} \vdots D]'J - 2J'[\mathbf{1} \vdots D] + (n+1)J \\
&= 4\left[\begin{array}{c|c} n+1 & \mathbf{1}'D \\ \hline D'\mathbf{1} & D'D \end{array}\right] - 2\left[\begin{array}{c|c} n+1 & \mathbf{1}'J \\ \hline D'\mathbf{1} & D'J \end{array}\right] - 2\left[\begin{array}{c|c} n+1 & \mathbf{1}'D \\ \hline J'\mathbf{1} & J'D \end{array}\right] + \left[\begin{array}{c|c} n+1 & (n+1)\mathbf{1}' \\ \hline (n+1)\mathbf{1} & (n+1)J \end{array}\right] \\
&= \left[\begin{array}{c|c} n+1 & -(n+1)\mathbf{1}' + 2\mathbf{1}'D \\ \hline -(n+1)\mathbf{1} + 2D'\mathbf{1} & (n+1)J - 2D'J - 2J'D + 4D'D \end{array}\right].
\end{aligned}
$$

4. Balanced saturated main effect plans. We now characterize balanced saturated main effect plans through D'D.

Theorem 4.1. *If a saturated main effect plan of the 2^n factorial is balanced then:*

$$
(4.1) \qquad D'D = \frac{(n + 1 - b)}{4} I + \frac{(n + 1 + 2a + b)}{4} J
$$

where a *and* b *are integers such that* $(n + 1 + a)/2$ *and* $(n + 1 + 2a + b)/4$ *are non-negative integers between* 0 *and* $n + 1$.

Proof. The definition of balance in (v) of section 2 implies that the information matrix is of the form:

$$X'X = \left[\begin{array}{c|c} n+1 & a\mathbf{1}' \\ \hline a\mathbf{1} & (n+1-b)I + bJ \end{array}\right] . \tag{4.2}$$

Hence, from (3.7) we have:

$$a\mathbf{1}' = -(n + 1)\mathbf{1}' + 2\mathbf{1}'D, \text{ or}$$

$$\mathbf{1}'D = \tfrac{1}{2}(a + n + 1)\mathbf{1}' . \tag{4.3}$$

Also from (3.7) we must have

$$(n + 1)J - 2D'J - 2J'D + 4D'D = (n + 1 - b)I + bJ.$$

Hence, using (4.3) we have

$$(n + 1)J - (a + n + 1)J - (a + n + 1)J + 4D'D = (n + 1 - b)I + bJ$$

so that

$$D'D = \frac{(n + 1 - b)}{4} I + \frac{(n + 1 + 2a + b)}{4} J . \tag{4.4}$$

Since a diagonal element of D'D is the squared length of a (0,1)-vector, it follows immediately that $\frac{(n + 1 - b)}{4} + \frac{(n + 1 + 2a + b)}{4} = (n + 1 + a)/2$ is an integer between 0 and n + 1. Finally, the innerproduct of two (0,1)-vectors is quite clearly an integer between 0 and n + 1 and hence (n + 1 + 2a + b)/4 is such a number. This completes the proof of the theorem.

The following theorem, which we give without proof, provides certain algebraic results concerning X'X and $(X'X)^{-1}$ of balanced saturated main effect plans.

<u>Theorem 4.2. The characteristic roots of X'X of a balanced saturated main effect plan of the 2^n factorial are:</u>

$$(4.5)\qquad \begin{cases} \lambda_1 = (n + 1 - b) \text{ with multiplicity } n - 1. \\ \lambda_2 = \left(2(n + 1) + (n - 1)b\right)/2 + \left((n - 1)^2b^2 + 4na^2\right)^{\frac{1}{2}}/2 \\ \lambda_3 = \left(2(n + 1) + (n - 1)b\right)/2 - \left((n - 1)^2b^2 + 4na^2\right)^{\frac{1}{2}}/2 \end{cases}$$

so that:

$$(4.6)\qquad \det X'X = (n + 1 - b)^{n-1}\left[(n + 1)^2 + (n^2 - 1)b - na^2\right]$$

$$(4.7)\qquad \det(X'X)^{-1} = (n + 1 - b)^{1-n}\left[(n + 1)^2 + (n^2 - 1)b - na^2\right]^{-1}$$

$$(4.8)\qquad \operatorname{trace}(X'X)^{-1} = (n - 1)\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}$$

Note that in this theorem we have assumed non-singularity of the fraction. Clearly singularity will occur if and only if

$$(n + 1 - b)^{n-1}\left[(n + 1)^2 + (n^2 - 1)b - na^2\right] = 0, \text{ i.e.,}$$

$$(4.9)\qquad a = \left\{ \frac{(n + 1)}{n} \left[(n + 1) + (n - 1)b\right]\right\}^{\frac{1}{2}} \text{ or } b = n + 1.$$

Thus, in the case of a singular balanced saturated main effect plan we have the characterization:

$$(4.10)\qquad D'D = \frac{(n + 1 + a)}{2} J$$

or

$$(4.11)\qquad D'D = \frac{(n + 1 - b)}{4} I + \frac{(n + 1) + 2\left\{\frac{n + 1}{n}\left[(n + 1) + (n - 1)b\right]\right\}^{\frac{1}{2}} + b}{4} J .$$

Hence we have a test of singularity for these designs.

5. Optimal balanced saturated main effect plans. Let us consider optimal plans with respect to the criterion det X'X, although one may use other criteria such as trace$(X'X)^{-1}$, max root$(X'X)^{-1}$, etc. as well. The reason for selecting the det X'X criterion is to tie up balanced saturated main effect plans with balanced incomplete block designs.

Carrying out the maximization of det X'X results in the solutions a = 0 and b = 0. Hence we have:

$$D'D = \frac{(n+1)}{4} I + \frac{(n+1)}{4} J \; . \tag{5.1}$$

This characterizes the class of Hadamard plans in complete agreement with the results reported by Raktoe and Federer [1970a]. Since a necessary condition for existence of these plans is that (n + 1) is divisible by 4 we may write n + 1 = 4m, so that (5.1) becomes:

$$D'D = mI + mJ \; . \tag{5.2}$$

As illustrations of the results obtained so far, consider the 2^3 factorial. Equation (4.4) implies that balanced saturated main effect plans must satisfy:

$$D'D = \frac{(4-b)}{4} I + \frac{(4+2a+b)}{4} J. \tag{5.3}$$

Clearly the designs:

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

are balanced because:

$$D_1'D_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = I + J = \frac{(4 - 0)}{4} I + \frac{(4 + 2(0) + 0)}{4} J \tag{5.5}$$

$$D_2'D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = \frac{(4 - 0)}{4} I + \frac{(4 + 2(-2) + 0)}{4} J \; . \tag{5.6}$$

However, note that D_1 is an optimal design since it satisfies (5.2), i.e.,

$$D_1'D_1 = \frac{(3 + 1)}{4} I + \frac{(3 + 1)}{4} J \; . \tag{5.7}$$

Thus using the det X'X criterion one would prefer D_1 over D_2.

6. Balanced optimal saturated main effect plans and v,k,λ configurations.
Following Ryser [1963] we define a v,k,λ configuration (or v,k,λ design) to be an arrangement of v elements into v sets such that each set contains exactly k distinct elements and such that each pair of sets has exactly λ elements in common, where $0 \le \lambda < k < v$. (Note, we are allowing λ to be equal to 0.) In design terminology a v,k,λ configuration is a balanced incomplete design with parameters v, $b = v$, k, $r = k$, and λ. The $v \times v$, (0,1)-incidence matrix A of a v,k,λ configuration satisfies the properties:

$$A'A = AA' = (k - \lambda)I + \lambda J \tag{6.1}$$

$$|\det A| = k(k - \lambda)^{\frac{1}{2}(v-1)}. \tag{6.2}$$

Now, let Q be a (0,1)-matrix of order v, containing exactly t +1's. Let $k = t/v$ and set $\lambda = k(k - 1)/(v - 1)$, with $0 \le \lambda < k < v$, then it follows from

Ryser's [1956] results that:

$$(6.3) \qquad |\det Q| \le k(k - \lambda)^{\frac{1}{2}(v-1)}$$

with equality holding if and only if Q is the incidence matrix of a v,k,λ configuration.

Now, consider designs which always include the treatment combination $(0\ 0 \cdots 0)$. Then it follows from (3.6) that:

$$(6.4) \qquad |\det X_{(n+1)\times(n+1)}| = 2^n \left|\det\left[\mathbf{1} \,\middle|\, \begin{matrix} 0' \\ \hline D^* \end{matrix}\right]\right| = 2^n |\det D^*|$$

where D^* is an $n \times n$ (0,1)-matrix. From Williamson [1946], Ryser [1956], or Raktoe and Federer [1970a] we know that:

$$(6.5) \qquad |\det D^*| \le 2^{-n}(n + 1)^{\frac{(n+1)}{2}}$$

with equality holding if and only if D^* is obtained from a Hadamard matrix, i.e., in the case of equality with $n + 1 = 4m$ we have:

$$(6.6) \qquad |\det D^*| = 2^{-(4m-1)}(4m)^{2m} .$$

Also, in the case where (6.6) holds it is known that D^* is the incidence matrix of a $v = 4\lambda - 1$ $(= n)$, $k = 2\lambda$, λ configuration.

Let us now see how (6.1), (6.3), and (6.4) can be used in saturated main effect plans. Let t denote the number of 1's in the matrix D^*, then for the 2^n factorial the range of t is:

$$(6.7) \qquad n \le t \le n^2 - n + 1.$$

For the $v = n$, $k = t/n$, $\lambda = (t/n)(t/n - 1)/(n - 1) = t(t - n)/n^2(n - 1)$ configuration to make sense, we must consider the values n, 2n, 3n, $\cdots$, (n - 1)n for t. Setting t = dn, d = 1, 2, $\cdots$, n - 1, we see that k = d, $\lambda = d(d-1)/(n-1)$. For a v = n, k = d, $\lambda = d(d-1)/(n-1)$ configuration to exist we must have that λ is a non-negative integer, i.e., d(d - 1) is divisible by (n - 1). As an illustration, the following table shows values of v,k,λ for v = n $\leq$ 7:

(6.8)

v	2	3	4	5
(k,λ)	(1,0)	(1,0), (2,1)	(1,0), (3,2)	(1,0), (4,3)

6	7	etc.
(1,0), (5,4)	(1,0), (3,1), (4,2), (6,5)	etc.

Clearly, λ is a non-negative integer <u>for all</u> n if t = n or t = (n - 1)n. In these cases we have the (n,1,0) and (n,n-1,n-2) configurations, respectively. Hence we have the balanced saturated main effect plans:

$$(6.9)\qquad \begin{bmatrix} 0' \\ \hline D^{++} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0' \\ \hline D^{**} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Using (6.3) we have at once the following:

<u>Theorem 6.1. For the 2^n factorial the balanced saturated main effect plan $\begin{bmatrix} 0' \\ \hline D^{++} \end{bmatrix}$ is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^{*} \end{bmatrix}, \mathbf{1}'D^{*}\mathbf{1} = n \right\}$ and the balanced saturated main effect plan $\begin{bmatrix} 0' \\ \hline D^{**} \end{bmatrix}$ is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^{*} \end{bmatrix}, \mathbf{1}'D^{*}\mathbf{1} = (n-1)n \right\}$.</u>

Note that in the classes specified by the theorem the plans (6.9) are unique and that they are characterized by:

(6.10) $$D^{++\prime}D^{++} = D^{++}D^{++\prime} = I \text{ and } D^{**\prime}D^{**} = D^{**}D^{**\prime} = I + (n-2)J\ .$$

Suppose now that $n + 1 = 4m$, i.e., $n + 1$ satisfies the necessary condition for X to be a Hadamard matrix. This implies that the range of t is:

(6.11) $$4m - 1 \le t \le (4m - 1)(4m - 2) + 1.$$

For the $v = 4m - 1$, $k = t/(4m - 1)$, $\lambda = t(t - 4m + 1)/(4m - 1)^2(4m - 2)$ configuration to make sense, we must have $t \in \{4m - 1, 2(4m - 1), \cdots, (4m - 2)(4m - 1)\}$. Let $t = q(4m - 1)$, with $q \in \{1, 2, \cdots, (4m - 2)\}$, then $k = q$, $\lambda = q(q - 1)/(4m - 2)$. Now, λ must be a non-negative integer, i.e., $q(q - 1)$ must be divisible by $4m - 2$. The only choices of q which satisfy this condition are $q_1 = 1$, $q_2 = 2m$, $q_3 = 2m - 1$, and $q_4 = 4m - 2$. From (6.3) and (6.6) we then have immediately the following theorem:

Theorem 6.2. For the 2^n factorial: (i) the balanced saturated main effect plan corresponding to the $v = 4m - 1$, $k = 1$, $\lambda = 0$ configuration is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^* \end{bmatrix},\ \mathbf{1}'D^*\mathbf{1} = 4m - 1 \right\}$, (ii) the balanced saturated main effect plan corresponding to the $v = 4m - 1$, $k = 2m$, $\lambda = m$ configuration is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^* \end{bmatrix},\ \mathbf{1}'D^*\mathbf{1} = 2m(4m - 1) \right\}$, (iii) the balanced saturated main effect plan corresponding to the $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$ configuration is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^* \end{bmatrix},\ \mathbf{1}'D^*\mathbf{1} = (2m - 1)(4m - 1) \right\}$, and (iv) the balanced saturated main effect plan corresponding to the $v = 4m - 1$, $k = 4m - 2$, $\lambda = 4m - 3$ configuration is optimal in the class $\left\{ \begin{bmatrix} 0' \\ \hline D^* \end{bmatrix},\ \mathbf{1}'D^*\mathbf{1} = (4m - 2)(4m - 1) \right\}$.

Note that in this theorem (i) and (iv) are special cases of theorem 6.1. The plan in (ii) corresponds to the Hadamard (v,k,λ) configuration augmented with $(0\ 0\ 0 \cdots 0\ 0)$. Hence if a Hadamard matrix exists with the elements in the first row consisting of -1's except the element in the first column and the first column consisting of +1's, then we have immediately an optimal balanced saturated main effect plan obtained by deleting the first column and setting the -1's to 0. Note also that among all possible $\binom{2^{4m-1}}{4m}$ plans this Hadamard plan is optimal, as it satisfies equation (5.2).

The balanced optimal saturated main effect plans of theorem 6.2 are characterized respectively by:

$$
(6.12)\qquad
\begin{aligned}
&\text{(i)} && D^{*\prime}D^{*} = D^{*}D^{*\prime} = I\\
&\text{(ii)} && D^{*\prime}D^{*} = D^{*}D^{*\prime} = mI + mJ\\
&\text{(iii)} && D^{*\prime}D^{*} = D^{*}D^{*\prime} = mI + (m-1)J\\
&\text{(iv)} && D^{*\prime}D^{*} = D^{*}D^{*\prime} = I + (4m-3)J
\end{aligned}
$$

To illustrate the results of this section consider the following saturated main effect plans of the 2^7 factorial:

$$
(6.13)\qquad
D_3 = \begin{bmatrix}
0&0&0&0&0&0&0\\
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1
\end{bmatrix}, \quad
D_4 = \begin{bmatrix}
0&0&0&0&0&0&0\\
1&0&0&1&0&1&1\\
1&1&0&0&1&0&1\\
1&1&1&0&1&0&0\\
0&1&1&1&0&0&1\\
1&0&1&1&1&0&0\\
0&1&0&1&1&1&0\\
0&0&1&0&1&1&1
\end{bmatrix},
$$

$$
D_5 = \begin{bmatrix}
0&0&0&0&0&0&0\\
0&1&0&1&0&1&0\\
1&0&0&1&1&0&0\\
0&0&1&1&0&0&1\\
1&1&1&0&0&0&0\\
0&1&0&0&1&0&1\\
1&0&0&0&0&1&1\\
0&0&1&0&1&1&0
\end{bmatrix}, \quad
D_6 = \begin{bmatrix}
0&0&0&0&0&0&0\\
0&1&1&1&1&1&1\\
1&0&1&1&1&1&1\\
1&1&0&1&1&1&1\\
1&1&1&0&1&1&1\\
1&1&1&1&0&1&1\\
1&1&1&1&1&0&1\\
1&1&1&1&1&1&0
\end{bmatrix}.
$$

As can be seen D_3 is optimal in the class $\left\{ \begin{bmatrix} 0' \\ - \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 7 \right\}$, D_4 is optimal in the class $\left\{ \begin{bmatrix} 0' \\ - \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 28 \right\}$, D_5 is optimal in the class $\left\{ \begin{bmatrix} 0' \\ - \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 21 \right\}$, and D_6 is optimal in the class $\left\{ \begin{bmatrix} 0' \\ - \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 42 \right\}$.

7. Complementary balanced optimal saturated main effect plans and v,k,λ configurations. Let $t = 4m - 1$ and consider the balanced optimal saturated main effect plans corresponding to the configurations (i) $v = 4m - 1$, $k = 1$, $\lambda = 0$, (ii) $v = 4m - 1$, $k = 2m$, $\lambda = m$, (iii) $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, and (iv) $v = 4m - 1$, $k = 4m - 2$, $\lambda = 4m - 3$. If we permute 0's to 1's and 1's to 0's in the incidence matrices of these configurations we obtain the complementary configurations ($\overline{\text{i}}$) $v = 4m - 1$, $k = 4m - 2$, $\lambda = 4m - 3$, ($\overline{\text{ii}}$) $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, ($\overline{\text{iii}}$) $v = 4m - 1$, $k = 2m$, $\lambda = m$, and ($\overline{\text{iv}}$) $v = 4m - 1$, $k = 1$, $\lambda = 0$. Clearly the configurations (i) to (iv) are closed under complementation (this last word meaning permuting 0's to 1's and 1's to 0's).

Consider the complementary saturated main effect plans obtained from the configurations ($\overline{\text{i}}$) to ($\overline{\text{iv}}$) and augmentation of $\mathbf{1}' = (1\ 1\ 1 \cdots 1\ 1)$. Equivalently, these designs are obtained by the map:

$$\tau: \begin{bmatrix} 0' \\ \hline D^* \end{bmatrix} \longrightarrow \begin{bmatrix} 1' \\ \hline \overline{D}^* \end{bmatrix} \tag{7.1}$$

where $\overline{D}^*$ is the matrix obtained from D^* by replacing 0's by 1's and 1's by 0's. In other words, if D^* is the incidence matrix of a v,k,λ configuration then $\overline{D}^*$ is the incidence matrix of the complementary v,k,λ configuration.

Paik and Federer [1970] and more recently Srivastava, Raktoe, and Pesotan [1971] (in a more general setting) have shown that if D is a saturated main effect plan and $\overline{D}$ is obtained from D by permuting the levels 0 and 1 of the factors, then the corresponding information matrices have the same determinant.

By invoking this invariance result mentioned at the beginning of the section, we have immediately:

Theorem 7.1. For the 2^n factorial: (i) the balanced complementary saturated main effect plan corresponding to the $v = 4m - 1$, $k = 4m - 2$, $\lambda = 4m - 3$ configuration is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = (4m - 2)(4m - 1) \right\}$, (ii) the balanced complementary saturated main effect plan corresponding to the $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$ configuration is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = (2m - 1)(4m - 1) \right\}$, (iii) the balanced complementary saturated main effect plan corresponding to the $v = 4m - 1$, $k = 2m$, $\lambda = m$ configuration is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 2m(4m - 1) \right\}$, and (iv) the balanced complementary saturated main effect plan corresponding to the $v = 4m - 1$, $k = 1$, $\lambda = 0$ configuration is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ D^* \end{bmatrix}, \mathbf{1}'D^*\mathbf{1} = 4m - 1 \right\}$.

The characterizations of these plans are

$$
(7.2) \qquad
\begin{aligned}
&\text{(i)} && \bar{D}^{*\prime}\bar{D}^* = \bar{D}^*\bar{D}^{*\prime} = I + (4m - 3)J \\
&\text{(ii)} && \bar{D}^{*\prime}\bar{D}^* = \bar{D}^*\bar{D}^{*\prime} = mI + (m - 1)J \\
&\text{(iii)} && \bar{D}^{*\prime}\bar{D}^* = \bar{D}^*\bar{D}^{*\prime} = mI + mJ \\
&\text{(iv)} && \bar{D}^{*\prime}\bar{D}^* = \bar{D}^*\bar{D}^{*\prime} = I
\end{aligned}
$$

or the designs $D = \begin{bmatrix} \mathbf{1}' \\ \bar{D}^* \end{bmatrix}$ are characterized by:

$$
(7.3) \qquad
\begin{aligned}
&\text{(i)} && D'D = I + (4m - 2)J \\
&\text{(ii)} && D'D = mI + mJ \\
&\text{(iii)} && D'D = mI + (m + 1)J \\
&\text{(iv)} && D'D = I + J
\end{aligned}
$$

Note that the balanced optimal saturated main effect plans in theorem 7.1 have an additional interpretation in terms of balanced incomplete block designs with unequal block sizes. The four optimal plans satisfying (7.3) are incidence matrices of the following family of balanced incomplete block designs:

$$
(7.4)\quad
\begin{array}{ll}
\text{(i)} & v = 4m - 1,\ b = 4m,\ k_1 = 4m - 1,\ k_2 = \cdots = k_{4m} = 4m - 2,\ \lambda = 4m - 3 \\
\text{(ii)} & v = 4m - 1,\ b = 4m,\ k_1 = 4m - 1,\ k_2 = \cdots = k_{4m} = 2m - 1,\ \lambda = m - 1 \\
\text{(iii)} & v = 4m - 1,\ b = 4m,\ k_1 = 4m - 1,\ k_2 = \cdots = k_{4m} = 2m,\ \lambda = m \\
\text{(iv)} & v = 4m - 1,\ b = 4m,\ k_1 = 4m - 1,\ k_2 = \cdots = k_{4m} = 1,\ \lambda = 0
\end{array}
$$

To illustrate the results consider the complementary designs for those given in (6.13):

$$
(7.5)\quad
\bar{D}_3 = \begin{bmatrix} 1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \\ 1&1&1&0&1&1&1 \\ 1&1&1&1&0&1&1 \\ 1&1&1&1&1&0&1 \\ 1&1&1&1&1&1&0 \end{bmatrix}, \qquad
\bar{D}_4 = \begin{bmatrix} 1&1&1&1&1&1&1 \\ 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&0&1&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \\ 1&1&0&1&0&0&0 \end{bmatrix},
$$

$$
\bar{D}_5 = \begin{bmatrix} 1&1&1&1&1&1&1 \\ 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 1&1&0&0&1&1&0 \\ 0&0&0&1&1&1&1 \\ 1&0&1&1&0&1&0 \\ 0&1&1&1&1&0&0 \\ 1&1&0&1&0&0&1 \end{bmatrix}, \qquad
\bar{D}_6 = \begin{bmatrix} 1&1&1&1&1&1&1 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}.
$$

As stated in theorem 7.1, $\bar{D}_3$ is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ \bar{D}^* \end{bmatrix},\ \mathbf{1}'\bar{D}^*\mathbf{1} = 42 \right\}$, $\bar{D}_4$ is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ \bar{D}^* \end{bmatrix},\ \mathbf{1}'\bar{D}^*\mathbf{1} = 21 \right\}$, $\bar{D}_5$ is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ \bar{D}^* \end{bmatrix},\ \mathbf{1}'\bar{D}^*\mathbf{1} = 28 \right\}$, and $\bar{D}_6$ is optimal in the class $\left\{ \begin{bmatrix} \mathbf{1}' \\ \bar{D}^* \end{bmatrix},\ \mathbf{1}'\bar{D}^*\mathbf{1} = 7 \right\}$. Also it is clear that $\bar{D}_3$, $\bar{D}_4$, $\bar{D}_5$, and $\bar{D}_6$ are incidence matrices of the balanced incomplete block designs with the parameters as specified by (7.4). Hence the characterizations as given in (7.3) are:

$$
(7.6)\quad
\begin{array}{rl}
\text{(i)} & D'D = I + 6J \\
\text{(ii)} & D'D = 2I + 2J \\
\text{(iii)} & D'D = 2I + 3J \\
\text{(iv)} & D'D = I + J
\end{array}
$$

Note that from Srivastava, Raktoe, and Pesotan [1971] it follows that the information matrices of the designs under theorem 6.2 and the complementary ones in theorem 7.1 are orthogonally similar. From the fractional replicate viewpoint and the optimality criteria based on the spectrum of the information matrix, the designs and their complementary ones are equivalent. However, physically these designs are quite different, so that choosing among, for example, D_3 and $\bar{D}_3$ can be done on the basis of a physical property or a function of this.

8. Discussion. In this paper we have explored the case $n + 1 = 4m$, i.e., the number of two level factors is equal to $4m - 1$. The results may be extended to other cases when $n + 1 \neq 4m$. To our knowledge, this is the first paper which shows how the number of +1's are important in classifying and characterizing balanced optimal plans. Work on the distribution of +1's in saturated main effect plans and their relation to values of $|X'X|$ has been started recently by Werner [1970]. She attempted to tie up the value of the determinant of $X'X$ with the number of ones. The results obtained herein apply to this problem in that for a given number of ones in D^* it is shown that when a v,k,λ configuration exists then the plan is optimal in the sense that the determinant of $X'X$ or $D^{*\prime}D^*$ is a maximum. When a v,k,λ configuration does not exist then one may study $(v, k, \lambda_1, \lambda_2, \cdots, \lambda_n)$ configurations (i.e., partially balanced configurations) for the various values of t (= the number of ones in a plan D^*). Proceeding in this manner, one may be able to determine the various values for the determinant of $X'X$ or $D^{*\prime}D^*$. The authors believe that the coming years will be rich ones for fractional replication.

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